Physics Challenge for Teachers and Students

Solution to the November 2016 *Challenge* **Boards of education**

*(*The problem adapted from *Russian Physics Olympiads*, S. Kozel, V. Slobodyanin, Eds., Verbum-M, Moscow, 2002)

Description of the System

Three very long boards are stacked upon each other, with respective masses $m_1 = m_2 = m_3 = m$. The position of the left ends of each are x_1, x_2, and x_3, respectively. Initially, they are positioned directly atop each other: $x_1 = x_2 = x_3 = 0$. They are at rest: $v_{10} = v_{20} = v_{30} = 0$. We may define their relative velocities: $v_{12} = v_1 - v_2$ and $v_{23} = v_2 - v_3$. Frictional forces are at work between the surfaces of the boards. Both the static and kinetic coefficients of friction between boards m_1 and m_2 are given by $\mu_{s12} = \mu_{k12} = \mu$. Both the static and kinetic coefficients of friction between boards m_2 and m_3 are given by $\mu_{s23} = \mu_{k23} = 2\mu$. Both the static and kinetic coefficients of friction between board m_3 and the ground are given by $\mu_{s3} = \mu_{k3} = 3\mu$.

First, an instantaneous impulsive force is applied to the top board m_1: $F_1 = mv_0\,\delta(t)$. As a result, m_1, and only m_1, starts to move with an initial speed v_0 to the right. After a time τ_1 (the stated problem describes this time as t, but I prefer to employ t as my dynamical time variable and $\{\tau_i\}$ for time constants), the entire system comes to rest.

Later, the system is restored to its initial state. Then an instantaneous impulsive force of the same magnitude as the first one is applied to m_3: $F_3 = mv_0\,\delta(t)$. As a result, m_3, and only m_3, starts to move with the same initial speed v_0 to the right. After a time τ_2, the entire system again comes to rest.

Statement of the Problem

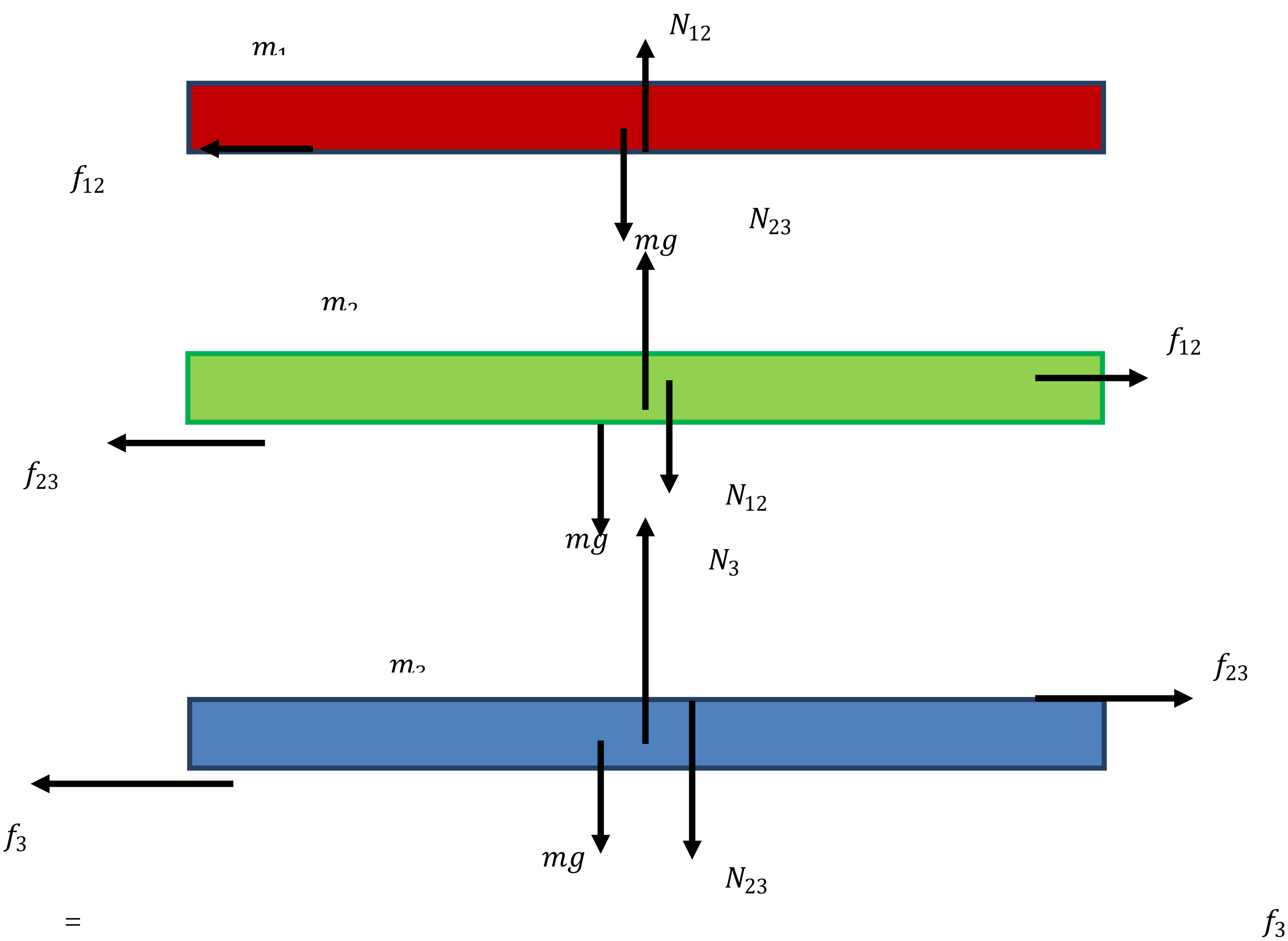
Find the value of τ_2 in terms of τ_1.

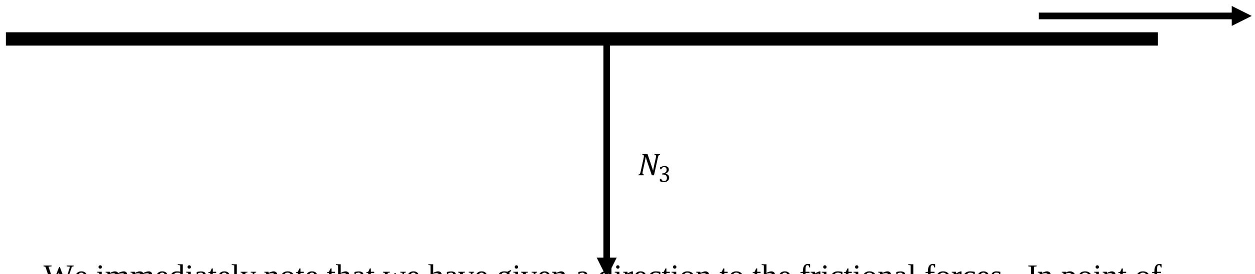
Proposed Solution

We will solve for τ_1 and τ_2 in turn, and then express τ_2 in terms of τ_1.

<u>General Considerations</u>

To do this we will start by drawing the free-body diagrams for the three boards. This will allow us to calculate normal and frictional forces as well as accelerations. Knowing the kinematical equations for uniform acceleration, we may write down the equations of motion for each of the three boards. We must break the motion down into steps, or regimes, when the frictional forces take different forms. Whenever there is relative motion, $v_{ij} \neq 0$, kinetic friction will operate: $|f_i| = \mu_{ki} N_i$. Whenever there is no relative motion, $v_{ij} = 0$, static friction is available to operate, producing as much force as is necessary to maintain the absence of relative motion, but not exceeding its maximum value, $|f_i| \leq \mu_{si} N_i$.





We immediately note that we have given a direction to the frictional forces. In point of fact, the direction will depend upon the relative velocities of the two surfaces in contact. We will accommodate this fact by ascribing a sign, $\pm$, to the frictional forces, in accord with the arrows as we have drawn them.

We proceed first to consider the vertical forces. All motion is in the horizontal, $x - direction$, and there is no motion in the vertical, $y - direction$. Therefore, there is no acceleration in the vertical: $a_{iy} = 0$. By Newton's second law, the sum of the vertical forces on any mass must therefore be zero, $\sum_k F_{iyk} = 0$.

$$\sum_k F_{1yk} = 0$$
$$N_{12} - mg = 0$$
$$N_{12} = mg$$

$$\sum_k F_{2yk} = 0$$
$$N_{23} - N_{12} - mg = 0$$
$$N_{23} - mg - mg = 0$$
$$N_{23} - 2mg = 0$$
$$N_{23} = 2mg$$

$$\sum_k F_{3yk} = 0$$
$$N_3 - N_{23} - mg = 0$$
$$N_3 - 2mg - mg = 0$$
$$N_3 - 3mg = 0$$
$$N_3 = 3mg$$

We may now make statements about the possible magnitudes of the frictional forces.

$$|f_{12}| \leq \mu_{s12} N_{12} = (\mu)(mg) = \mu mg, \text{ if } v_{12} = 0$$
$$|f_{12}| = \mu_{k12} N_{12} = (\mu)(mg) = \mu mg, \text{ if } v_{12} \neq 0$$

$$|f_{23}| \leq \mu_{s23} N_{23} = (2\mu)(2mg) = 4\mu mg, \text{ if } v_{23} = 0$$
$$|f_{23}| = \mu_{k23} N_{23} = (2\mu)(2mg) = 4\mu mg, \text{ if } v_{23} \neq 0$$

$$|f_3| \leq \mu_{s3} N_3 = (3\mu)(3mg) = 9\mu mg, \text{ if } v_3 = 0$$
$$|f_3| = \mu_{k3} N_3 = (3\mu)(3mg) = 9\mu mg, \text{ if } v_3 \neq 0$$

Case One

We now proceed to calculate the accelerations of the three masses for the first case.

Initial conditions:

$$v_1 = v_0$$
$$v_2 = 0$$
$$v_3 = 0$$

With relative velocities:

$$v_{12} = v_0 \neq 0$$
$$v_{23} = 0$$
$$v_3 = 0$$

Thus, if there is sufficient static friction, m_2 and m_3 may remain motionless, and the friction between m_1 and m_2 is necessarily kinetic and directed toward the left.

$$\sum\nolimits_k F_{1k} = ma_1$$
$$-f_{12} = ma_1$$
$$-\mu mg = ma_1$$
$$a_1 = -\mu g$$

We now assume that there is sufficient friction to maintain zero relative velocity between m_1 and $m_2$and check to see if the required frictional forces are less than the maximal static frictional forces.

$$\sum\nolimits_k F_{2k} = ma_2$$
$$f_{12} - f_{23} = ma_2$$
$$\mu mg - f_{23} = ma_2$$

We remember that $|f_{23}| \leq 4\mu mg$, if $v_{23} = 0$, and we know that $|f_{23}|$ is large enough to overcome f_{12}. If m_3 remains stationary, m_2 will remain stationary as well.

$$\sum\nolimits_k F_{3k} = ma_3$$
$$f_{23} - f_3 = ma_3$$

Since $|f_3| \leq 9\mu mg$, if $v_3 = 0$, and $|f_{23}| \leq 4\mu mg$ (whether static or kinetic), there is sufficient frictional force between m_3 and the ground to prevent m_3 from starting to move relative to the ground.

$$0 = ma_3$$
$$a_3 = 0$$

Since m_3 is stationary, it follows by our previous reasoning that m_2 will remain stationary as well.

Thus, only m_1 will move, slowing down with the negative acceleration we found, $a_1 = -\mu g$. From the kinematical equations, we find:

$$v_1 = v_0 - \mu g t$$

We may solve this equation for the time required for m_1, and thus the entire system, to come to rest:

$$0 = v_0 - \mu g \tau_1$$
$$\tau_1 = \frac{v_0}{\mu g}$$

Second Case

We may now proceed to the second case, in which it is the bottom board, m_3 which is struck. Nothing has changed about the forces, as we found them in the first case, except that the directions of friction and their magnitudes, if they are static, may change. The normal forces are completely unchanged.

These are the initial conditions for the second case:

$$v_1 = 0$$
$$v_2 = 0$$
$$v_3 = v_0$$

With relative velocities:

$$v_{12} = 0$$
$$v_{23} = -v_0$$
$$v_3 = v_0$$

Since there is motion between m_3 and the ground, we know that kinetic friction is at work between them. Similarly, we know that kinetic friction is at work between m_2 and m_3. Thus:

$$|f_{23}| = 4\mu m g$$
$$|f_3| = 9\mu m g$$

Since only m_3 is moving, and moving to the right, we may infer that both f_{23} and f_3, acting on m_3, are directed toward the left. Initially, $v_{12} = 0$. We need to find out if there is enough static frictional force on m_1 for it to accelerate at the same rate as m_2, $a_1 = a_2$. We start by assuming that they accelerate together, and then calculate the necessary static friction.

$$\sum_k F_{1k} = m a_1$$

$$-f_{12} = ma_1$$
$$-(-|f_{12}|) = ma_1$$
$$|f_{12}| = ma_1$$

$$\sum_k F_{2k} = ma_2$$
$$f_{12} - f_{23} = ma_2$$
$$-|f_{12}| - (-|f_{23}|) = ma_2$$
$$-|f_{12}| + |f_{23}| = ma_2$$
$$-|f_{12}| + 4\mu mg = ma_2$$
$$-|f_{12}| + 4\mu mg = ma_1$$

Substituting for ma_1 from our equation for m_1:

$$-|f_{12}| + 4\mu mg = |f_{12}|$$
$$2|f_{12}| = 4\mu mg$$
$$|f_{12}| = 2\mu mg$$

However, we already know that $|f_{12}| \leq \mu mg$. Therefore, we must be in the regime of kinetic friction:

$$|f_{12}| = \mu mg$$

$$|f_{12}| = ma_1$$
$$\mu mg = ma_1$$
$$a_1 = \mu g$$

$$-\mu mg + 4\mu mg = ma_2$$
$$3\mu mg = ma_2$$
$$a_2 = 3\mu g$$

$$\sum_k F_{3k} = ma_3$$
$$f_{23} - f_3 = ma_3$$
$$-4\mu mg - 9\mu mg = ma_3$$
$$-13\mu mg = ma_3$$
$$a_3 = -13\mu g$$

We may now invoke the kinematical equations to obtain equations for the velocities of the three boards:

$$v_1 = \mu gt$$
$$v_2 = 3\mu gt$$
$$v_3 = v_0 - 13\mu gt$$

Initially, m_3 is moving fastest and slowing down. Both m_1 and m_2 start from rest and are speeding up. However, m_2 is accelerating faster and gaining both speed and distance relative to m_1. Unless something changes, m_1 will never catch up to m_2.

The next event of interest will be when the velocities of m_2 and m_3 become equal. Then, there is a momentary possibility that they will start to move with the same velocity, should static friction be sufficient.

Let us calculate the time, t_1, at which this occurs.

$$v_2 = v_3$$
$$3\mu g t_1 = v_0 - 13\mu g t_1$$
$$16\mu g t_1 = v_0$$
$$t_1 = \frac{v_0}{16\mu g}$$

At this point, $v_{12} = v_1 - v_2 = \mu g t_1 - 3\mu g t_1 = 2\mu g t_1 = 2\mu g\left(\frac{v_0}{16\mu g}\right) = \left(\frac{1}{8}\right) v_0 \neq 0$, so we are in the regime of kinetic friction:

$$|f_{12}| = \mu m g$$

Similarly, $v_3 = v_0 - 13\mu g t_1 = v_0\left(\frac{16}{16}\right) - 13\mu g t_1\left(\frac{v_0}{16\mu g}\right) = \left(\frac{16}{16}\right) v_0 - \left(\frac{13}{16}\right) v_0 = \left(\frac{3}{16}\right) v_0 \neq 0$, so we are again in the regime of kinetic friction:

$$|f_3| = 9\mu m g$$

We now need to see if there is sufficient static friction between m_2 and m_3 for the two of them to move together, with the same acceleration, $a_2 = a_3$.

If they do move together, they will both be slowing down. The frictional force between them will act to the left on m_2 and to the right on m_3. Since m_1 is still moving slower than the two of them, the force of friction between m_1 and m_2 will continue be kinetic friction, directed to the right on m_1 and to the left on m_2.

Let us see what size of static friction, $|f_{23}|$, would be required between m_2 and m_3 for them to move together.

$$\sum_k F_{2k} = m a_2$$
$$f_{12} - f_{23} = m a_2$$
$$-|f_{12}| - |f_{23}| = m a_2$$
$$-\mu m g - |f_{23}| = m a_2$$

$$\sum_k F_{3k} = m a_3$$

$$f_{23} - f_3 = ma_3$$
$$|f_{23}| - |f_3| = ma_3$$
$$|f_{23}| - 9\mu mg = ma_3$$
$$|f_{23}| - 9\mu mg = ma_2$$

$$ma_2 = ma_3$$
$$-\mu mg - |f_{23}| = |f_{23}| - 9\mu mg$$
$$2|f_{23}| = 8\mu mg$$
$$|f_{23}| = 4\mu mg$$

Static friction is able to produce this much force ($|f_{23}| \leq 4\mu mg$). Therefore, we are in the regime of static friction between m_2 and m_3 and $|f_{23}| = 4\mu mg$.

We may now solve for the new accelerations of the three boards.

$$\sum_k F_{1k} = ma_1$$
$$-f_{12} = ma_1$$
$$-(-|f_{12}|) = ma_1$$
$$|f_{12}| = ma_1$$
$$\mu mg = ma_1$$
$$a_1 = \mu g$$

$$-\mu mg - |f_{23}| = ma_2$$
$$-\mu mg - 4\mu mg = ma_2$$
$$-5\mu mg = ma_2$$
$$a_2 = -5\mu g$$

$$4\mu mg - 9\mu mg = ma_3$$
$$-5\mu mg = ma_3$$
$$a_3 = -5\mu g$$

We may now write down the equations of motion for the three boards:

$$v_1 = \mu g t_1 + \mu g(t - t_1) = \mu g t_1 + \mu g t - \mu g t_1 = \mu g t$$

$$v_2 = 3\mu g\left(\frac{v_0}{16\mu g}\right) - 5\mu g(t - t_1) = \left(\frac{3}{16}\right)v_0 - 5\mu g t + 5\mu g t_1$$
$$= \left(\frac{3}{16}\right)v_0 - 5\mu g t + 5\mu g\left(\frac{v_0}{16\mu g}\right) = \left(\frac{8}{16}\right)v_0 - 5\mu g t + \left(\frac{5}{16}\right)v_0$$
$$= \left(\frac{8}{16}\right)v_0 - 5\mu g t = \left(\frac{1}{2}\right)v_0 - 5\mu g t$$

$$v_3 = \left(\frac{3}{16}\right) v_0 - 5\mu g(t - t_1) = \left(\frac{3}{16}\right) v_0 - 5\mu g t + 5\mu g t_1 = \left(\frac{3}{16}\right) v_0 - 5\mu g t + \left(\frac{5}{16}\right) v_0$$
$$= \left(\frac{1}{2}\right) v_0 - 5\mu g t$$

At this point, m_1 is speeding up, while m_2 and m_3 are slowing down. There will come a point when all three velocities are equal. At this point, one or more pairs of boards might be able to move together, if there is sufficient static friction.

Let us find the time, t_2, when all three are at the same speed. Since m_2 and m_3 are moving together, it is sufficient to find out when

$$v_1 = v_2$$
$$\mu g t_2 = \left(\frac{1}{3}\right) v_0 - 5\mu g t_2$$
$$6\mu g t_2 = \left(\frac{1}{2}\right) v_0$$
$$t_2 = \left(\frac{1}{12}\right)\left(\frac{v_0}{\mu g}\right)$$

After this event, frictional forces f_{12} on m_1, f_{23} on m_2, and f_3 on m_3 are devoted to slowing down the three moving boards and are therefore directed towards the left. By Newton's third law, f_{12} on m_2 and f_{23} on m_3 are directed towards the right. We must learn if there is enough frictional force for them all to move together, or any pair of them to move together.

$$\sum_k F_{3k} = ma_3$$
$$f_{23} - f_3 = ma_3$$
$$|f_{23}| - |f_3| = ma_3$$
$$|f_{23}| - 9\mu m g = ma_3$$

$$\sum_k F_{2k} = ma_2$$
$$f_{12} - f_{23} = ma_2$$
$$|f_{12}| - |f_{23}| = ma_2$$

$$\sum_k F_{1k} = ma_1$$
$$-f_{12} = ma_1$$
$$-|f_{12}| = ma_1$$

If

$$a_2 = a_3$$
$$ma_2 = ma_3$$
$$|f_{12}| - |f_{23}| = |f_{23}| - 9\mu m g$$
$$2|f_{23}| = |f_{12}| + 9\mu m g$$

$$|f_{23}| = \frac{1}{2}|f_{12}| + \frac{9}{2}\mu mg$$
$$|f_{23}| \geq \frac{9}{2}\mu mg$$
$$|f_{23}| > 4\mu mg$$

But we know $|f_{23}| \leq 4\mu mg$, so $a_2 = a_3$ is impossible. The friction between m_2 and m_3 is kinetic and $|f_{23}| = 4\mu mg$.

If

$$a_1 = a_2$$
$$ma_1 = ma_2$$
$$-|f_{12}| = |f_{12}| - |f_{23}|$$
$$2|f_{12}| = |f_{23}|$$
$$2|f_{12}| = 4\mu mg$$
$$|f_{12}| = 2\mu mg$$
$$|f_{12}| > \mu mg$$

But we know $|f_{12}| \leq mg$, so $a_1 = a_2$ is impossible. The friction between m_1 and m_2 is kinetic and $|f_{12}| = \mu mg$.

We may now solve for the accelerations of the three boards.

$$ma_1 = -|f_{12}|$$
$$ma_1 = -\mu mg$$
$$a_1 = -\mu g$$

$$ma_2 = |f_{12}| - |f_{23}|$$
$$ma_2 = \mu mg - 4\mu mg$$
$$ma_2 = -3\mu mg$$
$$a_2 = -3\mu g$$

$$ma_3 = |f_{23}| - 9\mu mg$$
$$ma_3 = 4\mu mg - 9\mu mg$$
$$ma_3 = -5\mu mg$$
$$a_3 = -5\mu g$$

Again, we may employ the kinematical equations to obtain the time dependence of the velocities of the three boards.

$$v_1 = \mu g t_2 - \mu g(t - t_2) = \mu g t_2 - \mu g t + \mu g t_2 = 2\mu g t_2 - \mu g t = 2\mu g\left(\left(\frac{1}{12}\right)\left(\frac{v_0}{\mu g}\right)\right) - \mu g t$$
$$= \left(\frac{1}{6}\right) v_0 - \mu g t$$

$$v_2 = \left(\frac{1}{2}\right) v_0 - 5\mu g t_2 - 3\mu g(t - t_2) = \left(\frac{1}{2}\right) v_0 - 5\mu g t_2 - 3\mu g t + 3\mu g t_2$$

$$= \left(\frac{1}{2}\right) v_0 - 2\mu g t_2 - 3\mu g t = \left(\frac{3}{3}\right)\left(\frac{1}{2}\right) v_0 - 2\mu g \left(\left(\frac{1}{12}\right)\left(\frac{v_0}{\mu g}\right)\right) - 3\mu g t$$

$$= \left(\frac{3}{6}\right) v_0 - \left(\frac{1}{6}\right) v_0 - 3\mu g t = \left(\frac{2}{6}\right) v_0 - 3\mu g t = \left(\frac{1}{3}\right) v_0 - 3\mu g t$$

$$v_3 = \left(\frac{1}{2}\right) v_0 - 5\mu g t_2 - 5\mu g(t - t_2) = \left(\frac{1}{2}\right) v_0 - 5\mu g t_2 - 5\mu g t + 5\mu g t_2 = \left(\frac{1}{2}\right) v_0 - 5\mu g t$$

All three boards start with the same velocity and have negative accelerations. Since the magnitudes of the the acceleration decrease from m_3 to m_2 to m_1, we expect them to come to a stop ($v = 0$) in the order m_3, then m_2, and then m_1.

Since m_3 will stop first, let us calculate t_3, when m_3 stops.

$$v_3 = 0$$

$$\left(\frac{1}{2}\right) v_0 - 5\mu g t_3 = 0$$

$$t_3 = \left(\frac{1}{10}\right)\left(\frac{v_0}{\mu g}\right)$$

We should check to see if there is sufficient static frictional force to keep m_3 motionless. Since $v_{23} \neq 0$, the frictional force between m_2 and m_3 is still kinetic and $|f_{23}| = 4\mu m g$.

$$m a_3 = |f_{23}| - |f_3|$$

$$m(0) = 4\mu m g - |f_3|$$

$$0 = 4\mu m g - |f_3|$$

$$|f_3| = 4\mu m g$$

$$|f_3| \leq 9\mu m g$$

So, there is sufficient static friction to keep m_3 motionless.

The frictional forces between m_2 and m_3 and between m_1 and m_2 remain kinetic. All the forces on m_1 and m_2 remain as they were, so the equations of motion for m_1 and m_2 remain the same.

The next event will be the stopping of m_2. Let us calculate t_4 when m_2 stops.

$$v_2 = 0$$

$$\left(\frac{1}{3}\right) v_0 - 3\mu g t_4 = 0$$

$$t_4 = \left(\frac{1}{9}\right)\left(\frac{v_0}{\mu g}\right)$$

We should check to see if there is sufficient static frictional force to keep m_3 and m_2 motionless. Since $v_{12} \neq 0$, the frictional force between m_1 and m_2 is still kinetic and $|f_{12}| = \mu mg$.

$$ma_2 = |f_{12}| - |f_{23}|$$
$$m(0) = \mu mg - |f_{23}|$$
$$0 = \mu mg - |f_{23}|$$
$$|f_{23}| = \mu mg$$
$$|f_{23}| \leq 4\mu mg$$

$$ma_3 = |f_{23}| - |f_3|$$
$$m(0) = \mu mg - |f_3|$$
$$0 = \mu mg - |f_3|$$
$$|f_3| = \mu mg$$
$$|f_3| \leq 9\mu mg$$

The required frictional forces are less than the maximal values that may be provided by static friction, so m_2 and m_3 will remain motionless.
The force upon m_1 remains the same (μmg to the left), so the equation of motion for m_1 remains the same.

We may now solve for t_5, the time when m_1 comes to rest.

$$v_1 = 0$$
$$\left(\frac{1}{6}\right) v_0 - \mu g t_5 = 0$$
$$t_5 = \left(\frac{1}{6}\right)\left(\frac{v_0}{\mu g}\right)$$

At time t_5, all three boards have come to rest. Therefore t_5 is the time when the system comes to rest, τ_2.

$$\tau_2 = t_5$$
$$\tau_2 = \left(\frac{1}{6}\right)\tau_1$$

To employ the language of the problem as stated:

If $\boldsymbol{t}$ is the time required for the system to come to rest after an impulsive force is exerted on the top board, $\boldsymbol{t}/\boldsymbol{6}$ would be the time required for the system to come to rest if the same impulsive force were to be exerted on the bottom board instead.

(Submitted by Thomas Olsen, Dar Al Uloom University, College of Medicin, Riyadh, Saudi Arabia)

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Guidelines for contributors:

- We ask that all solutions, preferably in Word format, be submitted to the dedicated email address ***challenges@aapt.org***. Each message will receive an automatic acknowledgment.
- If your name is—for instance—Bob Dylan, please name the file **"Dylan17Feb"** (do not include your first initial) when submitting the February 2017 solution.
- The subject line of each message should be the same as the name of the solution file.
- The deadline for submitting the solutions is the last day of the corresponding month.
- Each month, a representative selection of the successful solvers' names will be published in print and on the web.
- If you have a message for the Column Editor, you may contact him at ***korsunbo@post.harvard.edu***; however, please do not send your solutions to this address.

Many thanks to all contributors and we hope to hear from many more of you in the future!

Many thanks to all contributors; we hope to hear from many more of you in the fall. We also hope to see more submissions of the original problems – thank you in advance!

--Boris Korsunsky, Column Editor