**Physics Challenge for Teachers and Students**

Solution to the January, 2017 Challenge**, The January joints.**

I will make two assumptions in interpreting the wording of the problem. First, I will assume the spring has a relaxed length *l* equal to the diagonal of the square. That is, if *L* is the length of a rod, then $l = 2^{1/2} L$. If that were not true, the assembly would not adopt the shape of a square when it is initially "placed" (presumably without hands holding it in shape) on the table. Second, I will assume that when the problem asks about the "original position" of the mass, the intended position is that shown in the diagram in the problem statement, i.e., before *m* is pulled diagonally away from point A. In that case, the requested time *t* is one-quarter of a period, i.e., $t = \pi / (2\omega)$ where $\omega$ is the angular frequency for small oscillations of the system. (If instead the "original position" means the release point of the mass, then the requested time is a full period, four times longer than the answer I report here.)

It will be shown that $\omega = \sqrt{k/m}$ and thus the solution to the problem is

$$t = \boxed{\frac{\pi}{2}\sqrt{\frac{m}{k}}}. \tag{1}$$

Here are three methods of proving that $\omega$ has this form, in increasing order of complexity.

Method #1: By symmetry, as the two ends of the spring oscillate toward and away from the center of the square, the other two vertices of the square (to which *m* and A are attached) oscillate in the same way (relative to the center of the assembly) except for being 180° out of phase. (In other ways, when the spring ends move inward, the mass moves outward, and vice versa.) Therefore the motion would be unchanged if we instead connected the spring between *m* and point A. (As will be shown by the next two methods, it turns out that this statement is rigorously true only for *small* amplitudes of oscillation.) But then the rods no longer play a role. Discarding them, we simply have the standard oscillation of a mass *m* on a spring *k*, giving the angular frequency reported above.

Method #2: Consider an instant when the length of the spring is *x*, and the distance between *m* and point A is *y*. In other words, *y* measures the displacement of the mass from a fixed point on the table, and thus the time derivative of *y* is the velocity $\upsilon$ of the mass. At this instant, the assembly has the shape of a diamond with diagonals of length *x* and *y* so that

$$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = L^2 \quad \Rightarrow \quad y^2 = 4L^2 - x^2 \tag{2}$$

using the Pythagoras theorem. The time derivative of this result is

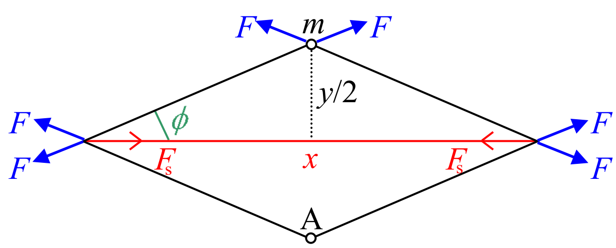
$$2y\upsilon = -2xV \tag{3}$$

where $V \equiv dx/dt$ is the rate of change of the length of the spring. For small amplitudes of oscillation, both $x$ and $y$ are always nearly equal to $l$, in which case Eq. (3) implies that $V \approx -\upsilon$. Defining the stretch of the spring to be $X \equiv x - l$ and noting that $V = dX/dt$, conservation of mechanical energy implies that the sum of the potential energy of the spring and the kinetic energy of the mass is at all times a constant $E$,

$$\tfrac{1}{2}kX^2 + \tfrac{1}{2}m\upsilon^2 = E \quad \Rightarrow \quad \tfrac{1}{2}kX^2 + \tfrac{1}{2}mV^2 \approx E\,. \tag{4}$$

But the final expression has exactly the same mathematical form as that of a mass $m$ on a spring $k$ and thus $\omega = \sqrt{k/m}$.

<u>Method #3:</u> Use the same symbols $x$, $y$, and $X$ as in Method #2. Suppose we consider an instant at which $X$ is positive, so the spring is stretched, as sketched below but with the distortion exaggerated for clarity.




The stretched spring applies inward forces $F_s$ on its two ends. Consequently there will be compressive forces $F$ along the rods, pushing outwards on each of their ends. (I have not shown the two forces at end A where the attachment force cancels them out.) The forces at each end of a rod must be equal and opposite, since the rods are massless. Furthermore since the joints are frictionless, we can assume there are no transverse forces. Balancing forces at either (massless) end of the spring, we have

$$2F\cos\phi = F_s \quad \Rightarrow \quad 2F\cos\phi = k(x-l) \tag{5}$$

in magnitude. Next, applying Newton's second law to mass $m$ with acceleration $a$, we find

$$F_{\text{net}} = ma \quad \Rightarrow \quad 2F\sin\phi = m\frac{d^2y}{dt^2}\,. \tag{6}$$

Divide Eq. (6) by (5) and use the fact that $\tan\phi = (y/2)/(x/2) = y/x$ to get

$$\omega^2 \frac{y(x-l)}{x} = \frac{d^2y}{dt^2}\,. \tag{7}$$

Substitute $x = l + X$ in both places on the left-hand side of this expression to get

$$\omega^2 \frac{yX}{l+X} = \frac{d^2y}{dt^2}\,. \tag{8}$$

Likewise substitute $x = l + X$ into Eq. (2) to get

$$y = \sqrt{4L^2 - l^2 - 2lX - X^2}\ . \tag{9}$$

Dropping the small $X^2$ term and substituting $L = 2^{-1/2} l$, Eq. (9) becomes

$$y \approx l\left(1 - \frac{2X}{l}\right)^{1/2} \approx l - X \tag{10}$$

using the binomial expansion in the second step. This result makes sense for small displacements from equilibrium: when *x* increases from *l* by *X*, then *y* decreases from *l* by *X*. But because it is merely approximate, Method #1 is only valid for small amplitudes. Substituting Eq. (10) into both places in Eq. (8), we how have

$$\varpi^2 \frac{(l - X)X}{l + X} \approx \frac{d^2(l - X)}{dt^2}\ . \tag{11}$$

Using the binomial expansion again, we get

$$\frac{1}{l + X} = \frac{1}{l}\left(1 + \frac{X}{l}\right)^{-1} \approx \frac{l - X}{l^2} \tag{12}$$

so that Eq. (11) becomes

$$\varpi^2 X\left(1 - \frac{X}{l}\right)^2 \approx -\frac{d^2X}{dt^2}\ . \tag{13}$$

Retaining only the lowest (linear) term in *X* on the left-hand side of this equation finally results in

$$\varpi^2 X \approx -\frac{d^2X}{dt^2} \tag{14}$$

which formally proves that the assembly exhibits simple harmonic motion at angular frequency $\varpi$.

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– The subject line of each message should be the same as the name of the solution file.

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Many thanks to all contributors and we hope to hear from many more of you in the future!
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